Deptt- MATHEMATICS

College- SOGHRA COLLEGE,BIHAR SHARIF

Part- BSc PART 2

① Solve: $p^2 - p(e^x + e^{-x}) + 1 = 0$

Soln:- $p^2 - pe^x - pe^{-x} + 1 = 0 \Rightarrow p(p - e^x) - e^{-x}(p - e^x) = 0$

$\Rightarrow (p - e^x)(p - e^{-x}) = 0 \Rightarrow p = e^x, \quad p = e^{-x}$

If $p = e^x$, then $\frac{dy}{dx} = e^x \Rightarrow dy = e^x dx \Rightarrow y = e^x + c$

If $p = e^{-x} \Rightarrow \frac{dy}{dx} = e^{-x} \Rightarrow y = -e^{-x} + c$

$\therefore$ complete solution is $(y - e^x + c)(y + e^{-x} - c) = 0$ Ans

② Solve $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0, \quad p = \frac{dy}{dx}$

Soln:- $x^2p^2 + xyp - 6y^2 = 0$

$\Rightarrow x^2p^2 + 3xyp - 2xyp - 6y^2 = 0 \Rightarrow px(px + 3y) - 2y(px + 3y) = 0$

$\Rightarrow (px + 3y)(px - 2y) = 0$

$\Rightarrow px + 3y = 0$ or $px - 2y = 0$

$\Rightarrow x\frac{dy}{dx} + 3y = 0$ or $x\frac{dy}{dx} - 2y = 0$

$\Rightarrow \frac{dx}{x} + \frac{3dy}{y} = 0$ or $\frac{dx}{x} - \frac{2dy}{y} = 0$

$\Rightarrow xy^3 = c$ or $\frac{x}{y^2} = c$

$\therefore$ solution is $(xy^3 - c)\left(\frac{x}{y^2} - c\right) = 0$

③ Solve $yp^2 + (x - y)p - x = 0$

Soln:- $yp^2 + xp - yp - x = 0$

$\Rightarrow yp(p - 1) + x(p - 1) = 0$

$\Rightarrow (p - 1)(yp + x) = 0 \Rightarrow p = 1, \; yp + x = 0$

$\Rightarrow \frac{dy}{dx} - 1 = 0$ or $y\frac{dy}{dx} + x = 0$

$\Rightarrow y - x = c$ or $y^2 + x^2 = c$

$\Rightarrow \therefore (y - x - c)(x^2 + y^2 - c) = 0$ Ans.

(4) Solve $p^3(x+2y)+3p^2(x+y)+(y+2x)p=0$

Soln:- $p[p^2(x+2y)+3p(x+y)+y+2x]=0$

$\Rightarrow p[x(p^2+3p+2)+y(2p^2+3p+1)]=0$

$\Rightarrow p[(p+1)(p+2)x+(p+1)(2p+1)y]=0$

$\Rightarrow p(p+1)(px+2py+2x+y]=0$

$\Rightarrow p=0, \quad p+1=0, \quad px+2py+2x+y=0$

$\Rightarrow \frac{dy}{dx}=0, \quad \frac{dy}{dx}+1=0 \quad (x+2y)\frac{dy}{dx}+2x+y=0$

$\quad xdy+ydx+2ydy+2xdx=0$

$\Rightarrow y=c, \quad x+y=c, \quad \Rightarrow xy+x^2+y^2=c$

$\therefore$ Solution is

$(y-c)(x+y-c)(x^2+y^2+xy-c)=0$ Ans

(5) Solve:- $p^2+2py\cot x=y^2$

Soln:- $p^2+2py\cot x+y^2\cot^2x=y^2+y^2\cot^2x$

$\Rightarrow (p+y\cot x)^2=y^2\operatorname{cosec}^2x$

$\Rightarrow p+y\cot x=\pm y\operatorname{cosec}x$

$\Rightarrow \frac{dy}{dx}+y\cot x=y\operatorname{cosec}x$ or $\frac{dy}{dx}+y\cot x=-y\operatorname{cosec}x$

$\Rightarrow \frac{dy}{y}=\frac{1-\cos x}{\sin x}$ or $-\frac{dy}{y}=\frac{1+\cos x}{\sin x}$

$\Rightarrow \frac{dy}{y}=\frac{\sin x}{1+\cos x}$ or $-\frac{dy}{y}=\frac{\sin x}{1-\cos x}$

$\Rightarrow \log y=-\log(1+\cos x)+\log c$ or $-\log y=\log(1-\cos x)+\log c$

$\Rightarrow y(1+\cos x)=c$ or $y(1-\cos x)=c$

$\therefore$ Solution is $y(1\pm\cos x)=c$ Ans

I :- Equation solvable for y :-

(1) Solve $y = 2px + p^4x^2$, $p = \frac{dy}{dx}$

Soln :- $y = 2px + p^4x^2$ —— (1)

Diff. w. r. to x,

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2p^4x + 4p^3x^2\frac{dp}{dx}$$

$$\Rightarrow p + 2x\frac{dp}{dx} + 2p^3x\left(p + 2x\frac{dp}{dx}\right) = 0$$

$$1 + 2p^3x = 0 \quad \text{—— (3)}$$

$$\Rightarrow p + 2x\frac{dp}{dx} = 0 \quad \text{or}$$

$$\Rightarrow 2\frac{dp}{p} + \frac{dx}{x} = 0$$

$$\Rightarrow p^2x = c \Rightarrow p^2 = \frac{c}{x}$$

Putting in (1)

$$y = 2px + c^2$$

$$\Rightarrow (y - c^2)^2 = 4p^2x^2$$

$$\Rightarrow (y - c^2)^2 = 4 \cdot \frac{c}{x} \cdot x^2$$

$\therefore (y - c^2)^2 = 4cx$ is the complete solution.

(2) Solve $y + px = x^4p^2$

Soln:- $y + px = x^4p^2$ —— (1)

Diff. w. r. to x,

$$p + p + x\frac{dp}{dx} = 4x^3p^2 + 2px^4\frac{dp}{dx}$$

$$\Rightarrow 2p + x\frac{dp}{dx} = 2px^3\left(2p + x\frac{dp}{dx}\right)$$

$$\Rightarrow 2p + x\frac{dp}{dx} = 0, \quad 1 - 2px^3 = 0$$

when $2p + x\frac{dp}{dx} = 0$, $\Rightarrow \frac{dp}{p} + 2\frac{dx}{x} = 0 \Rightarrow px^2 = c$ —— (2)

Eliminating p from (1) & (2),

$$y + \frac{c}{x^2} \cdot x = x^4 \cdot \frac{c^2}{x^4} \Rightarrow y = -\frac{c}{x} + c^2$$ is the req. solution.

(3) Solve $y = (1+p)x + p^2$, $p = \frac{dy}{dx}$

Soln:- $y = (1+p)x + p^2$ —— (1)

Diff. w.r.to x,

$$\cancel{\frac{dy}{dx}} = 1 + \cancel{p} + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

$$\Rightarrow \frac{dx}{dp} + x = -2p$$

$$I.F. = e^{\int dp} = e^p$$

$$\therefore xe^p = \int -2pe^p dp$$

$$\Rightarrow xe^p = -2[pe^p - e^p] + c$$

$$\Rightarrow x = -2(p-1) + ce^{-p} \quad —— (2)$$

(1) & (2) $\Rightarrow y = (1+p)[-2(p-1)+ce^{-p}] + p^2$.

$\Rightarrow y = c(p+1)e^{-p} + (2-p^2)$ is the req. soln.

(4) Solve $y = 2px + f(xp^2)$

Soln:- $y = 2px + f(xp^2)$

Diff. w.r.to x,

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + f'(xp^2)\left[p^2 + 2xp\frac{dp}{dx}\right]$$

$$\Rightarrow \left(p + 2x\frac{dp}{dx}\right)\left[1 + pf'(xp^2)\right] = 0$$

$$\Rightarrow p + 2x\frac{dp}{dx} = 0 \Rightarrow 2\frac{dp}{p} + \frac{dx}{x} = 0$$

$$\Rightarrow p^2x = c \Rightarrow p = \sqrt{c/x}$$

$\therefore y = 2\sqrt{cx} + f(c)$ is the required solution

II :- Equation solvable for x :-

① Solve for x : $y = 3px + 6p^2y^2$

Soln :- $y = 3px + 6p^2y^2$ ——— ①

$\Rightarrow 3x = \frac{y}{p} - 6py^2$

Diff. w.r.t. y,

$3\frac{dx}{dy} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 6y^2\frac{dp}{dy} - 12py$

$\Rightarrow (1 + 6p^2y)\left(2p + y\frac{dp}{dy}\right) = 0$

$\Rightarrow 2p + y\frac{dp}{dy} = 0 \Rightarrow py^2 = c \Rightarrow p = \frac{c}{y^2}$ ——— ②

① & ② $\Rightarrow y = 3\frac{c}{y^2}x + 6 \cdot \frac{c^2}{y^4}y^2$

$\Rightarrow y^3 = 3cx + 6c^2$

② ~~y~~ Solve for x : $y = 2px + y^2p^3$

Soln :- $y = 2px + y^2p^3$ ——— ①

$\Rightarrow 2x = \frac{y}{p} - p^2y^2$

Diff. w.r.t. y,

$2\frac{dx}{dy} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2p^2y - 2py^2\frac{dp}{dy}$

$\Rightarrow \frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2p^2y - 2py^2\frac{dp}{dy}$

$\Rightarrow \left(\frac{1}{p} + 2p^2y\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$

$\Rightarrow 1 + \frac{y}{p}\frac{dp}{dy} = 0 \Rightarrow \frac{dp}{p} + \frac{dy}{y} = 0$

$\Rightarrow py = c \Rightarrow p = \frac{c}{y}$ ——— ②

① & ② $\Rightarrow y = 2\frac{cx}{y} + y^2 \cdot \frac{c^3}{y^3}$

$\Rightarrow y^2 = 2cx + c^3$ is the req. soln.

Clairaut's Equation

P-⑥

Q: Solve and obtain the singular solution of following:

① $y = px + p^2$, $p = \frac{dy}{dx}$

Soln:- $y = px + p^2$ — ①

Diff. w.r.to x,

$\frac{dy}{dx} = \cancel{p} + x\frac{dp}{dx} + 2p\frac{dp}{dx}$ $\Rightarrow \frac{dp}{dx} = 0$ or $x + 2p = 0$

$\Rightarrow \frac{dp}{dx} = 0$ or $p = -\frac{x}{2}$ — ②

$\Rightarrow p = c$

$\therefore$ G.S:- $y = cx + c^2$

Eliminating p from ① & ②

$y = -\frac{x^2}{2} + \frac{x^2}{4} \Rightarrow y = -\frac{x^2}{4}$

$\Rightarrow x^2 = -4y$ is the req. singular solution

② $y = px + p - p^2$ — ①

Diff. w.r. to x

$\cancel{p} = \cancel{p} + x\frac{dp}{dx} + (1-2p)\frac{dp}{dx} = 0$

$\Rightarrow \frac{dp}{dx} = 0$ or $x + 1 - 2p = 0$

$\Rightarrow p = c$ $\Rightarrow p = \frac{x+1}{2}$ — ②

$\therefore$ G.S:- $y = cx + c - c^2$

① & ② $\Rightarrow y = \left(\frac{x+1}{2}\right)x + \frac{x+1}{2} - \left(\frac{x+1}{2}\right)^2$

$\Rightarrow y = \frac{x+1}{2}\left[x + 1 - \frac{x+1}{2}\right]$

$\Rightarrow y = \frac{(x+1)^2}{4}$

$\therefore 4y = (x+1)^2$ is the req. sing. soln.

③ $y = px + \frac{2}{p}$ — ①

Diff. w.r. to x,

$\cancel{p} = \cancel{p} + x\frac{dp}{dx} - \frac{2}{p^2}\frac{dp}{dx}$ $\Rightarrow$

$\Rightarrow \frac{dp}{dx} = 0$ or $x - \frac{2}{p^2} = 0 \Rightarrow p = \sqrt{\frac{2}{x}}$ — ②

$\Rightarrow p = c$

$\therefore$ G.S:- $y = cx + \frac{2}{c}$

from ① & ②, we have

$y = \sqrt{\frac{2}{x}} \cdot x + \sqrt{2x} = 2\sqrt{2x}$

$\Rightarrow y^2 = 8x$ is req. sing. soln.